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LINEAR PRESERVERS IN NONCLASSICAL CORRELATION THEORIES: AN INTRODUCTION

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Linear preserver classes used in recent quantum information science are briefly introduced. It has been fifteen years since linear positivity preservers that are not completely positivity preserving were employed in entanglement detection and quantification. Recently we have introduced the class of eigenvalue preservers that are not completely eigenvalue preserving to detect and quantify nonclassical correlation. Their concepts and an example of their usages are presented.

Keywords: Linear Preservers; Nonclassical Correlation

1. Introduction

Generally speaking, linear preservers are linear maps that preserve a certain property of a matrix, for which there have been much researches[1,2] performed by theorists of linear algebra. It was in 1996 that a class of linear preservers was introduced in quantum information science in the context of entanglement detection, which was a class of positivity-preserving but not completely positivity-preserving maps [or, positive but not completely positive (PnCP) maps].[3–5] The theory on PnCP maps has been developed in the are of entanglement theory[6] that has been growing rapidly and extensively.

Let us begin with the basics of entanglement to introduce the PnCP class. A quantum state of a bipartite AB is separable if and only if it is represented by a density matrix in the form of

$$\rho_{\mathrm{sep}}^{\mathrm{AB}} = \sum_i w_i \rho_i^{\mathrm{A}} \otimes \rho_i^{\mathrm{B}}$$

with nonnegative weights w_i satisfying $\sum_i w_i = 1$ and corresponding local density matrices ρ_i^{A} and ρ_i^{B}. Any state that is not separable is entangled. A PnCP map

$\Lambda_{\rm PnCP}:\mathbb{S}\rightarrow\mathbb{S}$ (here, $\mathbb{S}$ is the state space) is a map such that $\Lambda_{\rm PnCP}\rho\geq 0$ for all the density matrices ρ but $(I\otimes\Lambda_{\rm PnCP})\rho$ can be negative for some dimension of the identity map I. It is obvious that $(I^{\rm A}\otimes\Lambda^{\rm B}_{\rm PnCP})\rho^{\rm AB}_{\rm sep}\geq 0\quad\forall\rho^{\rm AB}_{\rm sep}$. Therefore, $\rho^{\rm AB}$ is entangled if $(I^{\rm A}\otimes\Lambda^{\rm B}_{\rm PnCP})\rho^{\rm AB}$ has negative eigenvalues. Note that this is a sufficient condition for a state to be entangled, but not a necessary condition, i.e., a partial PnCP map is not a perfect detection tool.

An often-used PnCP map is the transposition $\mathcal{T}$. It is known that $(I^{\rm A}\otimes\mathcal{T}^{\rm B})\rho^{\rm AB}<0$ is the necessary and sufficient condition for $\rho^{\rm AB}$ to be entangled when the dimension is 2×2, 2×3, or 3×2 but not for higher dimensions.[4,5] The partial transposition also provides a widely-used measure of entanglement, which is the negativity[7,8] defined as

$$N(\rho^{\rm AB})=-\sum_{\hat{e}<0}\hat{e}\in{\rm Eig}[(I^{\rm A}\otimes\mathcal{T}^{\rm B})\rho^{\rm AB}],\qquad(1)$$

where ${\rm Eig}[M]$ is the list of eigenvalues of a matrix M in which degenerate eigenvalues are listed without omission.

Besides PnCP, there has been no class of preservers actively studied until recently in quantum information science. It was indeed recently[9] that we have introduced the class of maps that are eigenvalue-preserving but not completely eigenvalue-preserving (EnCE) to detect and quantify nonclassical correlation that is defined differently from entanglement. It is designed to detect nonclassical correlation often known as quantum discord.[10] Quantum discord is the discrepancy of the values of two different representations of quantum mutual information, which is known[10] to vanish if and only if a state has a product eigenbasis, i.e., if and only if a state is represented as

$$\rho^{\rm AB}_{\rm pcc}=\sum_{ij}e_{ij}|u_i\rangle^{\rm A}\langle u_i|\otimes|v_j\rangle^{\rm B}\langle v_j|,\qquad(2)$$

where e_{ij} is the (ij)th eigenvalue; $\{|u_i\rangle^{\rm A}\}$ and $\{|v_j\rangle^{\rm B}\}$ are the eigenbases of the reduced density matrices for subsystems A and B, respectively. Such a state is called properly classically correlated (pcc).[11] A state that is not pcc is nonclassically correlated (ncc).

In this brief report, we introduce the basics and the recent results on EnCE maps in Sec. 2. A simple two-qubit example to compare nonclassical correlation with entanglement is shown in Sec. 3. Section 4 summarizes the report.

2. Preserver Class EnCE

An EnCE map $\Lambda_{\rm EnCE}:\mathbb{S}_d\rightarrow\mathbb{M}_d$ is a map such that it preserves the eigenvalues of a matrix while $I\otimes\Lambda_{\rm EnCE}$ does not necessarily for some dimension of I, where $\mathbb{S}_d$ is the space of $d\times d$ density matrices and $\mathbb{M}_d$ is the space of $d\times d$ matrices.

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Here, we do not impose the condition that the resultant matrix is Hermitian. Hermiticity-preserving (HP) EnCE maps were considered in Ref. 9 but later we moved to not-necessarily HP (nnHP) EnCE maps.[12]

As we showed in Ref. 12, a linear $\Lambda_{\rm EnCE}$ acting on $\rho \in \mathbb{S}_d$ should generate the biorthogonal sets $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ that contain the right and left eigenvectors from the eigenbasis $\{|u_i\rangle\}$ of ρ, i.e., $\sum_i e_i|u_i\rangle\langle u_i| \overset{\Lambda_{\rm EnCE}}{\mapsto} \sum_i e_i|a_i\rangle\langle b_i|$. Therefore, it is now obvious that partial EnCE maps $I^{\rm A}\otimes\Lambda^{\rm B}_{\rm EnCE}$ and $\Lambda^{\rm A}_{\rm EnCE}\otimes I^{\rm B}$ preserve the eigenvalues of a pcc state written as Eq. (2). Thus, one detects nonclassical correlation if a partial EnCE map changes eigenvalues of a bipartite state. Note that this is not a necessary condition for a state to be nonclassically correlated, i.e., a partial EnCE map is not a perfect detection tool. (Note added: Recently, a perfect detection method was developed by Chen *et al.*[13] Our method is still useful considering quantification.)

Unlike PnCP maps, we allowed nonlinear extension. This is because of the following theorem.[12]

> *Theorem:* For any nnHP linear EnCE map Υ and a bipartite system AB, the set of eigenvalues of $(I^{\rm A}\otimes\Upsilon^{\rm B})\rho^{\rm AB}$ is equal to that of $\rho^{\rm AB}$ for all $\rho^{\rm AB}\in\mathbb{S}_{d^{\rm A}d^{\rm B}}$ or equal to that of $(I^{\rm A}\otimes\mathcal{T}^{\rm B})\rho^{\rm AB}$ for all $\rho^{\rm AB}\in\mathbb{S}_{d^{\rm A}d^{\rm B}}$, where $\mathcal{T}$ is the matrix transposition.

Thus, as far as we use the changes in eigenvalues, we have only to consider the partial transposition among the partial maps of linear EnCE maps. As for nonlinear extension, see Ref. 9.

In a similar manner as negativity, one may introduce a quantification of nonclassical correlation of a bipartite state $\rho^{\rm AB}$ as[14]

$$F(\rho^{\rm AB}) = \sum_i |e_i - \tilde{e}_i|, \tag{3}$$

where e_i are the eigenvalues of $\rho^{\rm AB}$ and $\tilde{e}_i$ are the eigenvalues of $(I^{\rm A}\otimes\mathcal{T}^{\rm B})\rho^{\rm AB}$, both aligned in descending order. A drawback of this quantification is that it is not even subadditive with respect to a tensor product. We defined a more sophisticated measure named logarithmic fidelity[9] using linear and nonlinear EnCE maps, which is subadditive when an EnCE map is chosen appropriately.

3. Example

We have briefly reviewed the preserver class EnCE. Changes of eigenvalues due to a partial EnCE map are used to detect and quantify nonclassical correlation. This is in analogy to the usage of PnCP maps: negativeness after a partial PnCP map is used to detect and quantify entanglement.

Here, a simple and typical example is shown to depict the difference of nonclassical correlation from entanglement. As a mathematical nature, entanglement often suffers from so-called sudden death[15] under certain types of decoherence. This does not mean that nonclassical correlation suddenly disappear. Entanglement is defined as a correlation that cannot be produced from scratch by only local operations and classical communications.[16] It is not a suitable definition of quantumness in correlation when a state preparation stage is not of interest. At post-preparation stages, nonclassical correlation that we have discussed has more physically natural behavior than entanglement as we will see in the following example.

Consider the following noise operators acting on a couple of qubits with the noise parameter $0 \le p \le 1$:
(i) Depolarizing noise: $\mathcal{E}_{1,p} : \rho \mapsto (1-p)\rho + pI/4$.
(ii) Phase-dumping noise: $\mathcal{E}_{2,p} : \rho \mapsto (1-p)\rho + p\sum_{i=0}^{3}\langle i|\rho|i\rangle|i\rangle\langle i|$.
Let us set the initial state to $\rho_0 = |\psi_{\rm B}\rangle\langle\psi_{\rm B}|$ where $|\psi_{\rm B}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is one of the Bell states. We have

$$\mathcal{E}_{1,p}(\rho_0) = \begin{pmatrix} \frac{2-p}{4} & 0 & 0 & \frac{1-p}{2} \\ 0 & \frac{p}{4} & 0 & 0 \\ 0 & 0 & \frac{p}{4} & 0 \\ \frac{1-p}{2} & 0 & 0 & \frac{2-p}{4} \end{pmatrix}$$

which has the eigenvalues $p/4$ (with multiplicity three) and $(4-3p)/4$, and

$$\mathcal{E}_{2,p}(\rho_0) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1-p}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1-p}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

which has the eigenvalues 0 (with multiplicity two), $p/2$, and $(2-p)/2$. Let us apply $I \otimes \mathcal{T}$ to these resultant matrices. First,

$$(I \otimes \mathcal{T})\mathcal{E}_{1,p}(\rho_0) = \begin{pmatrix} \frac{2-p}{4} & 0 & 0 & 0 \\ 0 & \frac{p}{4} & \frac{1-p}{2} & 0 \\ 0 & \frac{1-p}{2} & \frac{p}{4} & 0 \\ 0 & 0 & 0 & \frac{2-p}{4} \end{pmatrix}$$

has the eigenvalues $(2-p)/4$ (with multiplicity three) and $(-2+3p)/4$ that is negative for $p < 2/3$. Second,

$$(I \otimes \mathcal{T})\mathcal{E}_{2,p}(\rho_0) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1-p}{2} & 0 \\ 0 & \frac{1-p}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

has the eigenvalues $1/2$ (with multiplicity two), $(1-p)/2$, and $(-1+p)/2$ that is negative for $p<1$.

It is now straightforward to calculate the negativity N found in Eq. (1) and the quantity F found in Eq. (3) as functions of p. We have

$$N[\mathcal{E}_{1,p}(\rho_0)] = -\min[0, (-2+3p)/4],$$
$$N[\mathcal{E}_{2,p}(\rho_0)] = (1-p)/2,$$

and

$$F[\mathcal{E}_{1,p}(\rho_0)] = F[\mathcal{E}_{2,p}(\rho_0)] = 2(1-p).$$

The behaviors of these quantities are illustrated in Fig. 1. As the system dimension

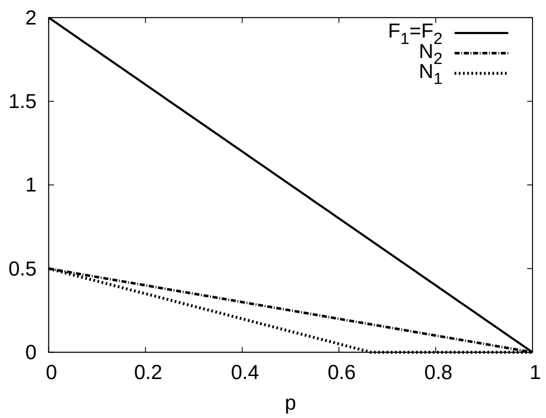

Fig. 1. Plots of $F[\mathcal{E}_{1,p}(\rho_0)]$ (equal to $F[\mathcal{E}_{2,p}(\rho_0)]$ in the present case) ($F_1 = F_2$), $N[\mathcal{E}_{2,p}(\rho_0)]$ (N_2), and $N[\mathcal{E}_{1,p}(\rho_0)]$ (N_1) as functions of the noise parameter p.

is 2×2, nonvanishing negativity is the necessary and sufficient condition for the system to possess entanglement. Entanglement vanishes at $p=2/3$ in case of depolarizing noise while it survives until $p=1$ in case of phase-dumping noise. In contrast, as for nonclassical correlation quantified by F, there is no difference in its behavior for the two cases.

The state affected by the depolarizing noise, $\mathcal{E}_{1,p}(\rho_0)$, is a pseudo-entangled state. It should possess a certain nonclassical correlation unless $p=1$. In this sense, quantification by F is more plausible than N for the present example.

4. Summary

Among the classes of linear preservers, we have briefly summarized the definition and the usage of the PnCP class in the context of entanglement detection and quantification, and have introduced the EnCE class recently developed in our contributions in the context of detection and quantification of nonclassical correlation defined differently from entanglement. The behaviours of nonclassical correlation and entanglement have been compared for a two-qubit system under the influence of depolarizing and phase-dumping noise.

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